Counting spanning trees with linear algebra

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Definitions

Definition 1 (Graph)

A simple undirected *graph* G is a pair (V, E), where V is a set and E is a symmetric subset of $V \times V \setminus \{(x,x), x \in V\}$. The elements of V are called the *vertices* of G and the elements of E are called the *edges* of G.

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Definition 2 (Path)

A *path* is a non-empty subgraph $P = (V_P, E_P)$ of the graph G of the form

$$V_P = \{x_0, x_1, \dots, x_k\} \qquad E_P = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\},$$

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Definition 3 (Connected graph)

A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G.

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A simple connected graph T is called *tree* if it is minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$.

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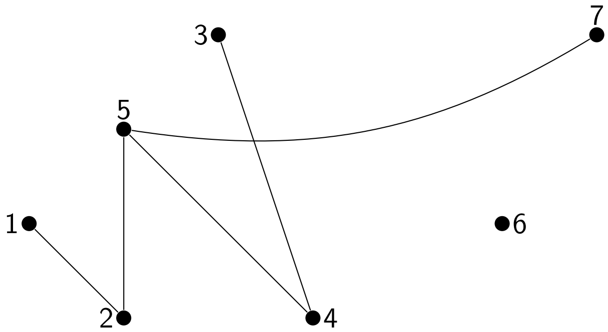
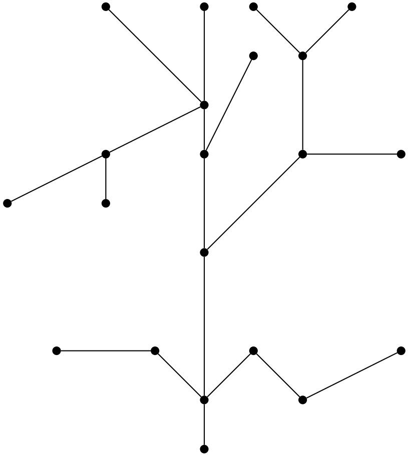
Definition 5 (Spanning tree)

If G is a connected graph, we say that T is a *spanning tree* of G if G and T have the same vertex set, and each edge of T is also an edge of G.



The graph on $V = \{1, \cdots, 7\}$ with edge set
$E = \{\{1, 2\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{5, 7\}\}$

Graph Visualization



Tree graph

Problem Statement

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You are given a finite simple connected graph G. How to calculate number of spanning trees of G?

Theorem

Theorem 6 (Matrix-Tree theorem)

Let U be a simple undirected graph. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of U. Define $(n-1) \times (n-1)$ matrix L_0 by

$$\ell_{ij} = \begin{cases} \text{the degree of } v_i \text{ if } i = j, \\ -1 \text{ if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are adjacent, and} \\ 0 \text{ otherwise} \end{cases}$$

where $1 \leq i, j \leq n-1$. Then U has exactly $\det L_0$ spanning trees.

Definition 7 (Matrix)

The matrix size $m \times n$ with real or complex entries is a rectangular array or table filled with real or complex numbers.

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$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

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Operations with matrices

- Addition
- Scalar multiplication
- Multiplication
- Transposing
- Inverting

Determinant

Definition 8 (Determinant of matrix)

Determinant of a square matrix is an antisymmetric multilinear function of the columns (or of the rows) of a matrix such that $\det I = 1$.

Properties of determinant

Properties

- $\det I = 1$
- Exchanging two rows (or two columns) reverses the sign of the determinant.
- The determinant is linear in each row (in each column) separately.
- For matrices of equal size X and Y: $\det XY = \det X \det Y$
- For matrix X of size $a \times a$ and constant $c \in \mathbb{C}$: $\det(cX) = c^a \det X$

Computing determinant: formula with permutations

Formula with permutations

$$\det A = \sum_{\pi \in Sym(n)} \text{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)},$$

where π ranges over the collection of all permutations of the set $\{1, 2, \dots, n\} = [n]$.

Row operations

Switching rows

$$\begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{m(n-1)} & a_{mn} \end{bmatrix} \hookrightarrow \begin{bmatrix} a_{m1} & \cdots & a_{m(n-1)} & a_{mn} \\ \cdots & \cdots & \cdots & \cdot \\ a_{11} & \cdots & a_{1(n-1)} & a_{1n} \end{bmatrix}$$

Multiplying row by a non-zero constant

$$\begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \hookrightarrow \begin{bmatrix} Ma_{11} & \cdots & Ma_{1(n-1)} & Ma_{1n} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Adding rows

$$\begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{m(n-1)} & a_{mn} \end{bmatrix} \hookrightarrow \begin{bmatrix} a_{11} + a_{m1} & \cdots & a_{1n} + a_{mn} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Computing determinant: Cofactor formula

Cofactor formula

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where $i \in [n]$ and C_{ij} equals $(-1)^{i+j}$ times determinant of $(n-1) \times (n-1)$ square matrix obtained by removing row i and column j. C_{ij} is called a *cofactor* of a_{ij}.

Example 9

Prove that the number of spanning trees of K_n is n^{n-2} (*Cayley's formula*).

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Proof.

$$L_0 = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \cdots & & & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

□

Example

Proof.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \cdots & & & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

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$$\det L_0 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & n \end{vmatrix} = n^{n-2}$$

Definition of a directed graph

Definition 9

Directed G graph is defined as follows: G=(V,E, s, t) where V and E are sets and s and t are the functions from E to V. For an edge e we think of s(e) as the starting vertex of e and t(e) is the ending vertex of e.

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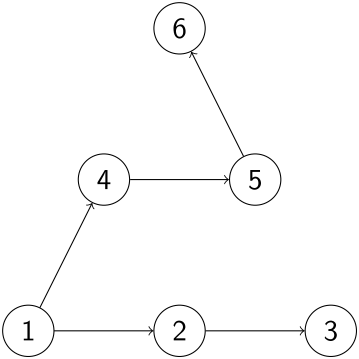

The case of directed graphs

Definition 10

Let G be a directed graph without loops. Let $\{v_1, v_2, \ldots, v_n\}$ be a verties of G, and let $\{e_1, e_2, \ldots, e_m\}$ denote the edges of G. Then the *incidence matrix* of G is $n \times m$ matrix A defined by

- $a_{ij} = 1$ if v_i is the starting vertex of e_j
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Theorem 11

Let G be a directed graph without loop, and let A be the incidence matrix of G. Remove any row of A and let A_0 be the remaining matrix. The number of spanning trees of G is $\det A_0 {A_0}^T$.